Verified Algorithm Analysis: Correctness and Complexity

A Biased Survey

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Focus on algorithm analyses in ITPs

Unless otherwise noted: in Isabelle/HOL

Please let me know of missed references

Out of scope: related work on completely automatic running time analyses by Martin Hofmann, Jan Hoffmann, Madhavan & Kuncak, ...

1. Mathematical Foundations

2. Programming and Verification Frameworks

3. Algorithms

Slides and results by Manuel Eberl

Classic concepts and results

- Landau symbols
- Generating functions
- Linear recurrences (theory and solver)
- Asymptotics of $n!$, Γ, H_n, C_n, $\ldots$

Akra–Bazzi theorem

Generalisation of the
Master Theorem for divide-and-conquer recurrences

Input (simple case):

$$T(x) = g(x) + \sum_{i=1}^{k} a_i T\left(\lfloor b_i x \rceil\right) \quad \text{for } g \in \Theta(x^q \ln^r x)$$

Result:

$$T \in \Theta(x^p) \quad T \in \Theta(x^p \ln \ln x)$$
$$T \in \Theta(x^q) \quad T \in \Theta(x^p \ln^{q+1} x)$$

where p is the unique solution to $\sum a_i b_i^p = 1$

Examples for Akra–Bazzi

Algorithm	Recurrence	Solution
Binary search	$T(\lceil n/2 \rceil) + O(1)$	$O(\log n)$
Merge sort	$T(\lfloor n/2 \rfloor) + T(\lceil n/2 \rceil) + O(n)$	$O(n \log n)$
Karatsuba	$2T(\lceil n/2 \rceil) + T(\lfloor n/2 \rfloor) + O(n)$	$O(n^{\log_2 3})$
Median-of-med's	$T(\lceil 0.2n \rceil) + T(\lceil 0.7n \rceil + 6) + O(n)$	$O(n)$

All of this is (almost) automatic.

Automated asymptotics

Isabelle can automatically prove

- $f(x) \xrightarrow{x \to L} L'$
- $f \in O(g)$, $f \in o(g)$, $f \in \Theta(g)$, $f(x) \sim g(x)$
- $f(x) \leq g(x)$ for x sufficiently close to L

for a wide class of $\mathbb{R}$-valued functions/sequences.

How? Multiseries expansions

Similar to algorithms used in Mathematica/Maple

Automated asymptotics

Example from Akra–Bazzi proof:

$$\lim_{x\to\infty} \left(1 - \frac{1}{b\log^{1+\varepsilon} x}\right)^p \left(1 + \frac{1}{\log^{\varepsilon/2}\left(bx + \frac{x}{\log^{1+\varepsilon} x}\right)}\right) - \left(1 + \frac{1}{\log^{\varepsilon/2} x}\right) \;=\; 0^+$$

Can be proved automatically in 0.3 s.

1. Mathematical Foundations

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For programming, refinement and verification
of algorithms
in Isabelle/HOL

Functional vs Imperative

Functional algorithms are expressed as HOL functions

Imperative algorithms are expressed in

`Imperative HOL`

a monadic framework with arrays and references by Bulwahn & Co [TPHOLs 08]

Can generate code in SML, OCaml, Haskell and Scala [Haftmann, N. FLOPS 10]

A problem:

Head-on verification of efficient algorithms
is painful or impossible

The cure:

Start from an abstract functional version
and refine it to an efficient (imperative) algorithm

A second problem:

Not every algorithm is deterministic:

```
for every neighbour do ...
```

Isabelle refinement framework

Lammich [ITP 12, ITP 13, ITP 15, CPP 16]

Provides abstract programming language with

- nondeterminism
- loops (incl. `foreach`)
- general recursion
- specification statement

Isabelle refinement framework

Lammich [ITP 12, ITP 13, ITP 15, CPP 16]

Stepwise program refinement by:

- algorithm refinement
- semi-automatic data refinement
  using verified collections library
- semi-automatic refinement to `Imperative HOL`

Almost all referenced Isabelle proofs can be found in the

Archive of Formal Proofs (AFP)

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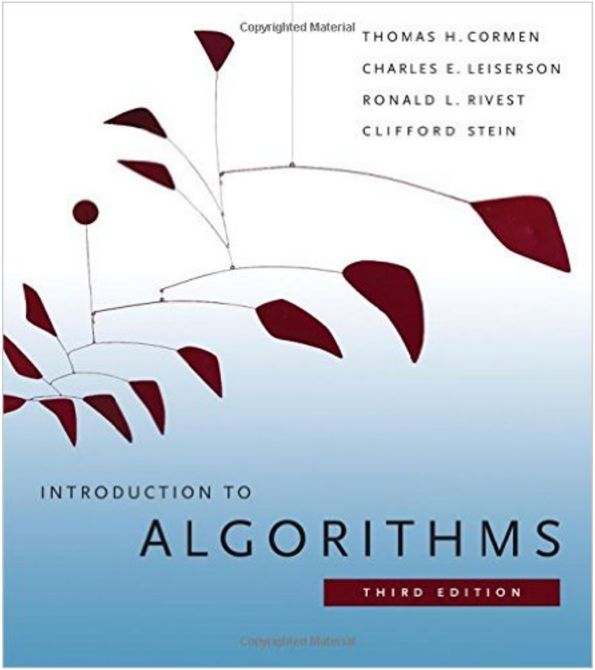
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3 Algorithms

Sorting & Order statistics

Search trees

Advanced Design and Analysis Techniques

Dynamic Programming

Advanced Data Structures

Graph Algorithms

Randomized Algorithms

Sorting

TIMsort: `java.util.Arrays.sort`

- A complex combination of mergesort and insertion sort on arrays
- De Gouw & Co [CAV 15] discover bug and suggest fixes
- De Gouw & Co [JAR 17] verify termination and exception freedom using the KeY system. Meanwhile: verification of functional correctness

k-th smallest element via median of medians

```
select k xs =
  let x = select ... (map median5 (chop 5 xs));
      (ls, es, gs) = partition3 x xs
  in if ... then select k ls
     else ... select ... gs
```

- Functional version by Eberl [AFP 17]
- Imperative refinement (incl linear time proof) by Zhan & Haslbeck [IJCAR 18] using Akra-Bazzi

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Popular case study for ITPs because nicely functional.

AVL and Red-Black trees:

- Filliâtre & Letouzey [ESOP 04] (in Coq)
- N. & Pusch [AFP 04]
- Krauss & Reiter [08]
- Charguéraud [10] (in Coq)
- Appel [11] (in Coq)
- Dross & Moy [14] (in SPARK)
- ...

Functional correctness

- Functional correctness obvious to humans
  but until recently more or less verbose in ITPs
- Most verifications based on some variant of
  $bst\langle l, a, r\rangle \leftrightarrow$
  $bst\ l \wedge bst\ r \wedge (\forall x \in l.\ x < a) \wedge (\forall x \in r.\ a < x)$
- Correctness proofs can be automated if $bst(t)$ is replaced by N. $sorted(inorder\ t)$ [N. ITP 16]
- Works for AVL, RBT, 2-3, 2-3-4, AA, splay and other search trees covered in this talk
- Not automated: balance invariants

Some more search trees
Not in CLRS

Weight-Balanced Trees

Nievergelt & Reingold [72,73]

- Parameter: balance factor $0 < \alpha \leq 0.5$
- Every subtree must be balanced:

$$\alpha \leq \frac{\text{size of smaller child}}{\text{size of whole subtree}}$$

- Insertion and deletion: single and double rotations depending on subtle numeric conditions
- Nievergelt and Reingold deletion incorrect
- Mistake discovered and corrected by
  Blum & Mehlhorn [80]
  and Hirai & Yamamoto [JFP 11] (in Coq)

Scapegoat trees

Anderson [89,99], Igal & Rivest [93]

Central idea:

Don't rebalance every time,
Rebuild when the tree gets "too unbalanced"

- Tricky: amortized logarithmic complexity analysis
- Recently verified [N. APLAS 17]

Functional finger tree

Hinze & Paterson [06]

Tree representation of sequences with

- access time to both ends in amortized $O(1)$
- concatenation and splitting in $O(\log n)$

General purpose data structure for implementing *sequences*, *priority queues*, *search trees*, ...

Verifications:

- Functional correctness:
  - Sozeau [ICFP 07] (in Coq)
  - Nordhoff, Körner, Lammich [AFP 10]
- Amortized complexity:
  - Danielsson [POPL 08] (in Agda)

❸ Algorithms

Sorting & Order statistics
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Huffman's algorithm

Huffman [52]

- Purpose: lossless text compression,
  eg Unix `zip` command
- Input: frequency table for all characters
- Output:
  variable length *binary code* for each character
  that minimizes the length of the encoded text
  $\Rightarrow$ short codes for frequent characters
- Functional correctness proof: Blanchette [JAR 09]

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The functional approach

Wimmer & Co [ITP 18]

Write recursive program

```
fib(n) = fib(n-1) + fib(n-2)
```

Crank the handle and obtain monadic memoized version

```
fib' n := do { a ← fib'(n-1);
               b ← fib'(n-2);
               return (a+b) }
```

with correctness theorem

```
snd (runstate (fib' n) empty) = fib n
```

where `f x := rhs` abbreviates

```
f x = do a ← lookup x;
         case a of
           Some r ⇒ return r |
           None ⇒ do r ← rhs;
                     update x r;
                     return r
```

Automation

- Automatic definition
  of monadic memoized function
- Automatic correctness proof
  via parametricity reasoning

How is the state $(=$ memory$)$ realized?

Two state monads

- Purely functional state monad
  based on some search tree
- State monad of `Imperative HOL` using arrays
  Same $O(.)$ running time
  as standard imperative programs

Applications

- Bellman-Ford (SSSP)
- CYK (Context-free parsing)
- Minimum Edit Distance
- Optimal Binary Search Tree
- ...

Including correctness proofs

But without complexity analysis (yet)

Optimal Binary Search Tree

Input:

- set of keys $k_1, \dots, k_n$
- access frequencies $b_1, \dots, b_n$ (hits):
  $b_i =$ number of searches for k_i
- and $a_0, \dots, a_n$ (misses):
  $a_i =$ number of searches in (k_i, k_{i+1})

Algorithms for building optimal search tree:

- Straightforward recursive cubic algorithm
- Knuth [71]: a quadratic optimization
- Yao [80]: simpler proof
- N. & Somogyi [AFP 18]

3. Algorithms
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   - **Advanced Data Structures**
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   - Randomized Algorithms

B-trees

Functional verification:

- Malecha & Co [POPL 10] (in Coq + Ynot)
- Ernst & Co [SSM 15] (in KIV)

Priority queues

Verification of functional implementations:

- Leftist heap
- Braun tree [N. AFP 14]
- Amortized analysis of
  Skew heap, Splay heap, Pairing heap
  N. [ITP 16], N. & Brinkop [JAR 18]
- Binomial heap and Skew binomial heap
  Meis, Nielsen, Lammich [AFP 10]

None of the above provide `decrease-key` ...
Challenge!

Union-Find

Charguéraud, Pottier, Guéneau [ITP 15, JAR 17, ESOP 18]

Framework ("Characteristic Formula"):

- Translates OCaml program into a logical formula that captures the program behaviour, including effects and running time.
- Import into Coq as axiom
- Verify program in Coq

Verified amortized complexity $O(\alpha(n))$ of each call
(Following Alstrup & Co [JA 14])

Strongly connected components

- Tarjan [72],
  verified by Schimpf & Smaus [ICLA 2015]
- Gabow [IPL 00],
  verified by Lammich [ITP 14]

Used in verified model checker CAVA

Dijkstra (SSSP)

Dijkstra [59]

Functional correctness verified:

- Nordhoff & Lammich [AFP 12]:
  purely functionally with finger trees
- Lammich [CPP 16]:
  imperative with arrays

Floyd-Warshall (APSP)

Functional correctness verified by Wimmer & Lammich [AFP 17]:

- Functional implementation
- Refined to imperative algorithm on an array
- Main complication: destructive update
- All related verifications make simplifying assumptions — also in CLRS

Maximum network flow

- Edmonds-Karp:
  Lammich & Sefidgar [ITP 16]
  Imperative, running time $O(|V||E|^2)$
- Push-Relabel (2 variants):
  Lammich & Sefidgar [JAR 17]
  Imperative, running time $O(|V|^2|E|)$

Competitive with a Java implementation

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Randomized algorithms formalized

Purely functionally via the Giry monad

Example:

$\texttt{do}\ \{\ \texttt{a} \leftarrow \textit{some distribution};$
$\quad\quad \texttt{b} \leftarrow \textit{some other distribution}\ (\texttt{a});$
$\quad\quad \texttt{return (a+b)}\ \}$

Quicksort

- van der Weegen & McKinna [ITP 08] (in Coq)
  Proved expected running time of randomized and deterministic quicksort $\leq 2n\lceil\log_2 n\rceil$
- Eberl & Co [ITP 18]
  Proved closed form $2(n+1)H_n - 4n$
  and asymptotics $\sim 2n\ln n$
- Tassarotti & Harper [ITP 18] (in Coq)
  Formalized and extended cookbook method for tail bounds [Karp JACM 94]
  Applied it to quicksort:
  $\Pr[T(n) > (c+1)n\log_{4/3} n + 1] \leq \frac{1}{n^{c-1}}$

Analysis of random BSTs

Eberl & Co [ITP 18]

"Random BST" means

BST generated from a random permutation of keys

Thm Expected height of random BST
$\leq \ldots \sim 3 \log_2 n$

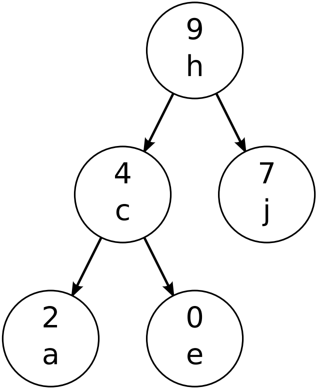
Thm Distribution of internal path lengths
$=$ distribution of running times of quicksort

Treaps

Aragon & Seidel [89, 96]

Random BSTs are pretty good,
but keys are typically not random

Treaps: combine each key with a random *priority*



treap = tree + heap

Treaps verified

Eberl & Co [ITP 18]

- Functional correctness straightforward
- Treaps need a *continuous* distribution of priorities to avoid duplicates (with probability 1)
- Reasoning about continuous distributions is hard because of measurability proofs
- **Thm** Distribution of treaps
  = distribution of random BSTs (modulo priorities)

Conclusion: Comparison with CLRS

The first 750 pages (parts I–VI, the "core")

- Much of the basic material has been verified
- Major omissions (afaik):
  - Hashing incl. probabilities
  - Fibonacci heaps
  - van Emde Boas trees